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REGULATOR PROBLEM FOR INDEX ONE DESCRIPTOR  
SYSTEMS

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# The Optimal Linear Quadratic Feedback State Regulator Problem for Index One Descriptor Systems

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**Abstract** In this note we present both necessary and sufficient conditions for the existence of a linear static state feedback controller if the system is described by an index one descriptor system. A priori no definiteness restrictions are made w.r.t. the quadratic performance criterium. It is shown that in general the set of solutions that solve the problem constitutes a manifold. This feedback formulation of the optimization problem is natural in the context of differential games and we provide a characterization of feedback Nash equilibria in a deterministic context.

**Keywords:** linear quadratic optimal control, descriptor systems, static stabilizing state feedback control.

**Jel-codes:** C61, C72, C73.

## 1 Introduction

In this note we consider the following problem: find a static stabilizing state feedback controller  $u(t) = Fx(t)$  that minimizes for all initial states the performance criterion

$$J := \int_0^\infty \{x^T(t)Qx(t) + u^T(t)Ru(t)\}dt \quad (1)$$

subject to the dynamics

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0. \quad (2)$$

Here,  $x \in \mathbb{R}^n$  is the state of the system and  $u \in \mathbb{R}^m$  the control.

Problems of this kind naturally appear in studying systems which operate under different timescales

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like e.g. in mechanical engineering where an electrically driven robot manipulator typically has slow mechanical dynamics and fast electrical dynamics, or in environmental economics where the global warming is assumed to be a system which has slow dynamics that is affected by various processes that have fast dynamics. Furthermore descriptor systems sometimes naturally appear in modeling systems like e.g. the Leontieff model in economics describing the relationship between the levels of production of a number of interrelated production sectors.

This problem, without the requirement that the control should be of a feedback type, has been considered by many authors. One of the first who considered this control problem was Pandolfi [16]. His results were later on generalized by e.g. Cobb [6] who gave both necessary and sufficient conditions under which the regular definite control problem has a solution in terms of a transformed system. Cheng et al. [5] used a similar approach to present a sufficient condition under which the problem has a solution. Bender et al. showed amongst other things in [1] that there is a linear variety of feedback gains, all of which yield the same minimal cost of (1). In particular they showed that there are various Riccati equations yielding the same solution. In Wang et al. [19] a parametrization of the optimal feedback gains was used to show that by an appropriate choice of the feedback gain for single input systems certain (LQR/unmodeled dynamics) robustness properties can be achieved. Further important, in particular numerical, results were obtained by Mehrmann and his coworkers (see e.g. [15], [14] and [2]). Katayama et al. [12] presented a sufficient condition in terms of a Riccati equation formulated in the original state parameters under which the problem has a solution. The singular control problem (i.e.  $R \geq 0$ ) for descriptor systems was analyzed by e.g. Geerts and Zhu et al. in [10], [22], whereas the  $H^2$  output feedback control problem was considered in Takaba et al. [18]. For multi-person games Xu et al., see e.g. [20, 21], derived some first results for descriptor systems, in particular for zero-sum games assuming a closed-loop perfect state information structure. Engwerda et al. [8] recently solved the general problem for index one systems assuming an open-loop information structure.

Based on an equivalence result in linear-quadratic theory we derive in Section III for the general indefinite linear-quadratic control problem a both necessary and sufficient condition for the existence of a stabilizing feedback control. In general this feedback matrix is not uniquely determined (see [1]). We give a parametrization of all feedback matrices that solve the problem. This result extends that of [19] where for a special class of systems just a partial specification of all these matrices was given (see Remark 3.5, below). In Section IV we use this equivalence result to characterize all linear feedback Nash equilibria in the corresponding differential game setting.

In an example we show how the freedom in the choice of the feedback matrix can be used by the player(s) to achieve some additional desired system properties. Section V concludes, whereas Section II states some notation and preliminary results.

## 2 Preliminaries

In this section we present some results that are used in the next section to prove our main result. First, consider the system

$$E\dot{x}(t) = Ax(t), \quad x(0) = x_0, \quad (3)$$

where  $\text{rank}(E) = r < n$ . System (3) (or the matrix pair  $(E, A)$ ) is called regular if  $\det(\lambda E - A) \neq 0$ . From [9] we recall the so-called Weierstrass canonical form.

**Theorem 2.1** *If (3) is regular, then there exist nonsingular matrices  $X$  and  $Y$  such that*

$$Y^T E X = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \text{ and } Y^T A X = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix}, \quad (4)$$

where  $A_1$  is a matrix in Jordan form,  $N$  is a nilpotent matrix also in Jordan form and  $I$  is the identity matrix.  $A_1$  and  $N$  are unique up to permutation of Jordan blocks.  $\square$

We will assume that the degree of nilpotency of matrix  $N$  is one (i.e.  $N^1 = 0$ ), or, stated differently, system (3) has index one<sup>1</sup>. Under this assumption, with

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} := X^{-1}x, \text{ where } x_1 \in \mathbb{R}^r \text{ and } x_2 \in \mathbb{R}^{n-r} \quad (5)$$

and

$$B_1 := [I_r \ 0]Y^T B \text{ and } B_2 := [0 \ I_{n-r}]Y^T B, \quad (6)$$

the state feedback problem (1,2) can be rewritten as follows: minimize w.r.t.  $F$

$$J = \int_0^\infty \{ [x_1^T(t) \ x_2^T(t)] X^T (Q + F^T R F) X \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \} dt \quad (7)$$

subject to

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \left( \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} F X \right) \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}; \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = X^{-1}x_0. \quad (8)$$

Of course problem (7,8) is not completely specified, as we did not specify the set of admissible feedback strategies yet. Throughout this note we restrict the analysis to the set of linear state feedback controls that stabilize the system for all consistent initial states. Recall that an initial state  $x_0$  in (3) is called consistent if with this choice of the initial state the system has a solution. The set of all consistent initial states of (3) is given by  $\{x_0 \mid x_0 = X[\bar{x}_1^T \ 0]^T, \bar{x}_1 \in \mathbb{R}^r\}$ . As was shown in [11] this set is independent of the choice from matrix  $X$  chosen in the Weierstrass canonical form. A necessary and sufficient condition for the existence of such a matrix is to assume that system (2) (or  $(E, A, B)$ ) is finite dynamics stabilizable<sup>2</sup>. Notice that this property holds if and only if  $\text{rank}[\lambda E - A \ B] = n$  for all  $\lambda \in \mathbb{C}_0^+$ <sup>3</sup>. Furthermore we restrict in the analysis to the case that the closed-loop system  $E\dot{x}(t) = (A + BF)x(t)$  has no impulsive modes or, stated differently,  $(E, (A + BF))$  has index one. So the system must be controllable at infinity (or, impulse controllable). That is, all impulsive modes of (2) can be transformed into finite dynamic modes using static state feedback control<sup>4</sup>. These requirements lead then to the assumption that  $F \in \mathcal{F}$ , where

$$\mathcal{F} := \{F \mid \text{all finite eigenvalues of } (E, A + BF) \text{ are stable and } (E, A + BF) \text{ has index one}\}.$$

We now have the following elementary property.

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<sup>1</sup>This is equivalent to the assumption that  $\text{rank}([E \ AW]) = n$ , where the image of matrix  $W$  equals the null space of  $E$ . Notice furthermore that in that case  $\det(\lambda E - A)$  is not a constant.

<sup>2</sup>Formal: (2) is finite dynamics stabilizable if there exists a feedback  $u(t) = Fx(t)$  such that all finite eigenvalues of the system  $E\dot{x}(t) = (A + BF)x(t)$  are stable.

<sup>3</sup> $\mathbb{C}_0^+$  is the set of complex numbers with non-negative real part.

<sup>4</sup>Notice that system (2) is impulse controllable if and only if  $\text{rank}[E \ AW \ B] = n$ .

**Lemma 2.2** Assume  $(E, A)$  is regular and has index one. Then for all  $F \in \mathcal{F}$

1.  $G := I + B_2 F X_2$  is invertible;
2.  $(E, A + BF)$  is regular.

**Proof:**

1. To show that matrix  $G$  is invertible we first note that, with  $W \in \mathbb{R}^{n \times n}$  a matrix which image equals the null space of  $E$  and thus has rank  $n - r$  (see footnote 1),

$$\begin{aligned} \text{rank}(\begin{bmatrix} E & (A + BF)W \end{bmatrix}) &= \text{rank}(Y^T \begin{bmatrix} E & (A + BF)W \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}) \\ &= \text{rank}(\begin{bmatrix} Y^T E X & Y^T (A + BF) X X^{-1} W X \end{bmatrix}). \end{aligned}$$

Since  $EW = 0$  or, equivalently,  $Y^T E X X^{-1} W X = 0$  we have that  $X^{-1} W X = \begin{bmatrix} 0 & 0 \\ W_1 & W_2 \end{bmatrix}$ , where matrix  $[W_1 \ W_2]$  has full row rank  $n - r$ . Consequently,

$$\text{rank}(\begin{bmatrix} E & (A + BF)W \end{bmatrix}) = \text{rank}(\begin{bmatrix} I_r & 0 & B_1 F [W_1 \ W_2] \\ 0 & 0 & G [W_1 \ W_2] \end{bmatrix}).$$

So obviously, this matrix has full row rank only if matrix  $G$  is invertible.

2. First notice that  $(E, A + BF)$  is regular if and only if  $\det(Y^T(\lambda E - (A + BF))X) \neq 0$ . Now

$$\begin{aligned} \det(Y^T(\lambda E - (A + BF))X) &= \det(\lambda \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A_1 + B_1 F X_1 & B_1 F X_2 \\ B_2 F X_1 & G \end{bmatrix}) \\ &= \det(\begin{bmatrix} \lambda I_r - (A_1 + B_1 F X_1) + B_1 F X_2 G^{-1} B_2 F X_1 & 0 \\ -B_2 F X_1 & -G \end{bmatrix}) \\ &= (-1)^{n-r} \det(G) \det(\lambda I_r - A_1 - B_1 F X_1 + B_1 F X_2 G^{-1} B_2 F X_1). \end{aligned}$$

Since by assumption  $F \in \mathcal{F}$ , by item 1,  $G$  is invertible. So we conclude that the system is regular.

□

Furthermore we recall from, e.g., [4, p.97] the next property.

**Lemma 2.3** Assume  $C \in \mathbb{R}^{n \times m}$  and  $D \in \mathbb{R}^{m \times n}$ . Then the following holds:

1.  $I_n + CD$  is invertible if and only if  $I_m + DC$  is invertible.
2. If  $I_n + CD$  is invertible then:  $C(I_m + DC)^{-1} = (I_n + CD)^{-1}C$ . □

For notational convenience the notation  $S := BR^{-1}B^T$  is used. In solving problem (1,2), with  $E = I$ , the next algebraic Riccati equation (ARE)

$$A^T K + KA - KSK + Q = 0. \tag{9}$$

plays an important role. A solution  $K$  of this equation is called *stabilizing* if matrix  $A - SK$  is stable. It is well-known that such a solution, if it exists, is unique. From [7, Theorem 5.14] (see also [3]) we recall

**Theorem 2.4** Consider problem (1,2) with  $E = I$ ,  $Q$  symmetric and  $R$  positive definite ( $R > 0$ ). Assume that  $(A, B)$  is stabilizable and  $u = Fx$ , with  $F \in \mathcal{F}_0 := \{F \mid A + BF \text{ is stable}\}$ . The linear quadratic control problem (1,2) has a minimum  $\hat{F} \in \mathcal{F}_0$  for  $J(F)$  for each  $x_0$  if and only if the algebraic Riccati equation (9) has a symmetric stabilizing solution  $K =: K^*$ . If this linear quadratic control problem has a solution, then the solution is uniquely given by  $\hat{F} = -R^{-1}B^TK^*$  and the optimal control in feedback form is

$$u^*(t) = -R^{-1}B^TK^*x^*(t) \text{ where } \dot{x}^*(t) = (A - SK^*)x^*(t); x^*(0) = x_0. \quad (10)$$

Moreover,  $J(u^*) = x_0^TK^*x_0$ .  $\square$

By using a pre-state feedback  $u = -R^{-1}V^Tx + v$  one can easily show that the above Theorem 2.4 specializes for the next problem (11) as formulated in Theorem 2.5.

$$\min_{F \in \mathcal{F}_0} J(F) := \int_0^\infty \{x^T(t)[I \ F^T] \begin{bmatrix} Q & V \\ V^T & R \end{bmatrix} \begin{bmatrix} I \\ F \end{bmatrix} x(t)\} dt \text{ subject to } \dot{x}(t) = (A + BF)x(t). \quad (11)$$

**Theorem 2.5** Consider problem (11) with  $R > 0$  and  $(A, B)$  stabilizable.

The linear quadratic control problem (11) has a minimum  $\hat{F} \in \mathcal{F}_0$  for  $J(F)$  for each  $x_0$  if and only if the algebraic Riccati equation

$$(A - BR^{-1}V^T)^TK + K(A - BR^{-1}V^T) - KSK + Q - VR^{-1}V^T = 0 \quad (12)$$

has a symmetric stabilizing solution  $K =: K^*$ .

If this linear quadratic control problem has a solution, then the solution is uniquely given by  $\hat{F} = -R^{-1}(V^T + B^TK^*)$  and the optimal control is

$$u^*(t) = -R^{-1}(V^T + B^TK^*)x^*(t) \text{ where } \dot{x}^*(t) = (A - BR^{-1}V^T - SK^*)x^*(t); x^*(0) = x_0. \quad (13)$$

Moreover,  $J(u^*) = x_0^TK^*x_0$ .  $\square$

### 3 Main Result

Theorem 3.1 presents the main result for the one player case. It serves also as a basis for the derivation of the result for the multi-player case.

**Theorem 3.1** Assume  $(E, A)$  is regular and has index one;  $(E, A, B)$  is finite dynamics stabilizable and  $Q, R$  are symmetric. Consider the notation from the introduction. Let  $X =: [X_1 \ X_2]$ , with

$$X_1 \in \mathbb{R}^{n \times r} \text{ and } X_2 \in \mathbb{R}^{n \times (n-r)}, \text{ and } X^T Q X =: \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}, \text{ where } Q_{11} \in \mathbb{R}^{r \times r}, Q_{12} \in \mathbb{R}^{r \times (n-r)}$$

and  $Q_{22} \in \mathbb{R}^{(n-r) \times (n-r)}$ .

Assume, moreover,  $\bar{R} := R + B_2^T Q_{22} B_2 > 0$ .

The LQ state feedback descriptor problem (1,2) has a solution  $u = Fx$ , with  $F \in \mathcal{F}$ , for all consistent initial states if and only if the next algebraic Riccati equation (14) has a symmetric stabilizing solution  $K =: K^*$  (that is:  $\sigma(A_1 + B_1 \bar{R}^{-1} B_2^T Q_{12}^T - B_1 \bar{R}^{-1} B_1^T K^* \subset \mathbb{C}^-)$ ) and the nonlinear equation (15) has a solution  $F =: F^*$ .

$$0 = (A_1 + B_1 \bar{R}^{-1} B_2^T Q_{12}^T)^TK + K(A_1 + B_1 \bar{R}^{-1} B_2^T Q_{12}^T) - KB_1 \bar{R}^{-1} B_1^T K + Q_{11} - Q_{12} B_2 \bar{R}^{-1} B_2^T Q_{12}^T \quad (14)$$

$$-\bar{R}^{-1}(-B_2^T Q_{12}^T + B_1^T K^*) = F(I + X_2 B_2 F)^{-1} X_1. \quad (15)$$

If the above equations have a solution,  $F^*$  solves the problem and the minimal cost is  $J^* = x_1^T K^* x_1$ , where  $x_1 := [I \ 0] X^{-1} x_0$ .

**Proof:** From (8) it follows that for an arbitrary  $F \in \mathcal{F}$

$$\dot{x}_1(t) = A_1 x_1(t) + B_1 F X_1 x_1(t) + B_1 F X_2 x_2(t) \quad (16)$$

$$0 = x_2(t) + B_2 F X_1 x_1(t) + B_2 F X_2 x_2(t), \quad (17)$$

where, by assumption,  $(A_1, B_1)$  is stabilizable.

By Lemma 2.2 matrix  $G := I + B_2 F X_2$  is invertible. Therefore we have from (17) that  $x_2(t) = -G^{-1} B_2 F X_1 x_1(t)$ . Substitution of this into (7) and (8) shows then that the optimization problem can be rewritten as the minimization of

$$J = \int_0^\infty \{x_1^T(t) [I - X_1^T F^T B_2^T G^{-T}] X^T (Q + F^T R F) X \begin{bmatrix} I \\ -G^{-1} B_2 F X_1 \end{bmatrix} x_1(t)\} dt \quad (18)$$

$$\text{subject to the system } \dot{x}_1(t) = (A_1 + B_1 F X_1 - B_1 F X_2 G^{-1} B_2 F X_1) x_1(t). \quad (19)$$

Next introduce  $\tilde{G} := I + X_2 B_2 F$ . By Lemma 2.3:  $X_2 G^{-1} = \tilde{G}^{-1} X_2$ . So,  $I - X_2 G^{-1} B_2 F = I - \tilde{G}^{-1} X_2 B_2 F = I - \tilde{G}^{-1}(\tilde{G} - I) = \tilde{G}^{-1}$ . Using this, (19) can be rewritten as  $\dot{x}_1(t) = (A_1 + B_1 F(I - \tilde{G}^{-1} X_2 B_2 F) X_1) x_1(t) = (A_1 + B_1 F \tilde{G}^{-1} X_1) x_1(t)$ . Now, let  $\tilde{F} := F(X_1 - X_2 G^{-1} B_2 F X_1) = F \tilde{G}^{-1} X_1$ . Then, using Lemma 2.3 again,  $B_2 \tilde{F} = B_2 F \tilde{G}^{-1} X_1 = (I + B_2 F X_2)^{-1} B_2 F X_1 = G^{-1} B_2 F X_1$ . Using this, the optimization in (18) can be rewritten as the minimization of

$$J = \int_0^\infty x_1^T(t) \{ [I - \tilde{F}^T B_2^T] X^T Q X \begin{bmatrix} I \\ -B_2 \tilde{F} \end{bmatrix} + \tilde{F}^T R \tilde{F} \} x_1(t) dt$$

$$\text{subject to } \dot{x}_1(t) = (A_1 + B_1 \tilde{F}) x_1(t). \quad (20)$$

It is easily verified that  $J$  can be rewritten as

$$J = \int_0^\infty x_1^T(t) \{ [I \ \tilde{F}^T] \begin{bmatrix} Q_{11} & -Q_{12} B_2 \\ -B_2^T Q_{12}^T & R + B_2^T Q_{22} B_2 \end{bmatrix} \begin{bmatrix} I \\ \tilde{F} \end{bmatrix} \} x_1(t) dt. \quad (21)$$

With  $Q := Q_{11}$ ,  $V := -Q_{12} B_2$ ,  $R := \bar{R}$ ,  $A := A_1$  and  $B := B_1$  in Theorem 2.5 it follows that the minimization of (21) subject to (20) has a solution  $\tilde{F}$  if and only if (14) has a stabilizing solution  $K^*$ . Furthermore, the optimal feedback matrix equals  $\tilde{F} = -\bar{R}^{-1}(-B_2^T Q_{12}^T + B_1^T K^*)$ . Since by definition  $\tilde{F} = F \tilde{G}^{-1} X_1$ , we conclude that problem (1,2) has a solution  $F$  if and only if additionally this equation (i.e. (15)) is solvable.  $\square$

### Remark 3.2

1. As already mentioned before, if (14) has a stabilizing solution  $K^*$  then it is unique. Therefore, to verify the solvability conditions, the most logical approach seems to verify first whether (14) has an appropriate solution, and next verify the solvability of (15). Now let for  $H \in \mathbb{R}^{m \times n}$

$$M(H) := -\bar{R}^{-1}(-B_2^T Q_{12}^T + B_1^T K^*)(X_1^T X_1)^{-1} X_1^T + H(I - X_1(X_1^T X_1)^{-1} X_1^T).$$

Since  $X_1$  is full column rank, it follows then that  $F$  solves (15) if and only if there exists a matrix  $H$  such that  $F(I + X_2B_2F)^{-1}$  satisfies  $F(I + X_2B_2F)^{-1} = M(H)$ .

Or, stated differently, equivalent to the statement that (15) has a solution  $F$  is that the matrix equation  $(I - M(H)X_2B_2)F = M(H)$  has a solution for some matrices  $F$  and  $H$ . In general the feedback solution  $F^*$  will not be uniquely determined. Example 3.4 illustrates this point.

2. Notice that Theorem 3.1 a priori just requires that matrix  $\bar{R}$  is positive definite. This implies that for certain types of problems it is not required in the original problem formulation that matrix  $R$  should be positive (semi) definite to conclude that the optimization problem has a solution.
3. Mark that the cost just depends on the initial state related to the dynamic part of the system.

□

A disadvantage of the above result is that one first has to perform some matrix transformations before one can draw conclusions. Below we present a sufficient condition in terms of the original model parameters to conclude that the problem has a solution. Following [12, 17] consider the matrix equation

$$LA + A^TK + Q - LBR^{-1}B^TK = 0, \text{ where } LE = E^TK. \quad (22)$$

and the feedback control

$$u(t) = -R^{-1}B^TKx(t) =: Fx(t), \text{ where } E\dot{x}(t) = (A - BR^{-1}B^TK)x(t), x(0) = x_0. \quad (23)$$

Then we have the following result. The proof is in the Appendix.

**Proposition 3.3** *Let  $L$ ,  $K$  and  $F$  solve (22,23). Using the notation of Theorem 3.1 assume that  $\sigma(A_1 + B_1F(I + X_2B_2F)^{-1}X_1) \subset \mathbb{C}^-$  and  $L_1 := X_1^TLY^{-T} \begin{bmatrix} I \\ 0 \end{bmatrix}$  is symmetric<sup>5</sup>. Then  $F$  is an optimal solution for the  $LQ$  feedback descriptor problem. The corresponding minimal cost is  $x_0^T LEx_0$ . □*

The next example illustrates the main result. The example is inspired by systems which operate under different time scales. We consider a simplistic optimal maintenance model where both variables with fast dynamics and slow dynamics occur.

**Example 3.4** Consider the optimal maintenance of a machine. Assume that there is a regular yearly update of the machine. On the other hand there is a daily routine machine inspection to make sure that the machine will operate properly during the year. Consider the next model:

$$E\dot{W}(t) = -\delta EW(t) - CD(t) + u_l(t) \quad (24)$$

$$C\dot{D}(t) = \frac{1}{\epsilon}(\alpha CD(t) - u_f(t)) + \beta EW(t). \quad (25)$$

Here (24) models the economic value ( $EW$ ) of the machine. It is assumed that this value depreciates over time but that its value can be increased again by either long-term maintain services or investments ( $u_l$ ).  $CD$  models the economic damage if there is a breakdown of the machine during the

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<sup>5</sup>It is easily verified that this assumption is, e.g., satisfied if we impose the additional restriction that  $L = K^T$  in (22)



year<sup>6</sup>. These cost grow very fast if there is a breakdown. This is modeled by the term  $\frac{\alpha}{\epsilon}CD$  on the right-hand side of (25), where  $\epsilon$  is assumed to be small. Furthermore it is assumed that the average economic damage due to breakdowns of the machine during every year is a fraction of the economic wealth of the machine (modeled by  $\beta EW$ ). Finally the term  $u_f$  in (25) models the daily inspection cost of the machine. The objective function of the owner of the machine is given by

$$J = \int_0^\infty \{EW^2(t) - \gamma CD^2(t) - \theta_l u_l^2(t) - \frac{\theta_f}{\theta + (1-\theta)\epsilon} u_f^2(t)\} dt. \quad (26)$$

This utility function expresses that the owner likes to maximize the economic wealth of the machine. However, a breakdown of the machine during the year should be avoided due to its negative impact on the production process. Furthermore, of course, investment and repairing is costly. The fact that the daily inspection cost should be low compared to the yearly long-term maintenance cost is expressed by the weight  $\frac{1}{\theta + (1-\theta)\epsilon}$  for the daily cost. Here  $\theta$  expresses that there is some lower-bound for daily operating cost. Furthermore, the total cost for repair every year should be a fraction of the economic value of the machine. This can be expressed by choosing relative high numbers of the weights  $\theta_i$  attached to the control variables compared to the weight one for the economic value of the machine. This enforces a moderate use of the control variables compared to the other variables. The fact that these control variables are quadratically penalized might be motivated by the observation that the larger these variables are probably the larger the repairing time will be, which will be disliked by the customers of the product. Given this model the owner likes to design an optimal repair service strategy.

With  $\epsilon = 0$  this problem can be rewritten, with  $x^T := [EW \ CD]$  and  $u^T := [u_l \ u_f]$ , as the minimization of (1,2) with  $Q := \begin{bmatrix} -1 & 0 \\ 0 & \gamma \end{bmatrix}$ ,  $R := \begin{bmatrix} \theta_l & 0 \\ 0 & \frac{\theta_f}{\theta} \end{bmatrix}$ ,  $E := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $A := \begin{bmatrix} -\delta & -1 \\ 0 & \alpha \end{bmatrix}$  and  $B := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ . Obviously, for practical applications, the defection rate is an important piece of information in the decision making about the optimal repair strategy. For that reason it seems to make sense to minimize  $J$  with respect to the set of state feedback controls  $u(t) = Fx(t)$ . With  $Y^T := \frac{-1}{\alpha^2} \begin{bmatrix} \alpha & 1 \\ 0 & -\alpha \end{bmatrix}$  and  $X := \begin{bmatrix} -\alpha & 0 \\ 0 & 1 \end{bmatrix}$ ,  $Y^T E X = E$  and  $Y^T A X = \begin{bmatrix} -\delta & 0 \\ 0 & 1 \end{bmatrix}$ .

Consequently,  $B_1 = \frac{1}{\alpha^2}[-\alpha \ 1]$ ,  $B_2 = [0 \ \frac{-1}{\alpha}]$ ,  $A_1 = -\delta$ ,  $X^T Q X = \begin{bmatrix} -\alpha^2 & 0 \\ 0 & \gamma \end{bmatrix}$  and  $\bar{R} = \begin{bmatrix} \theta_l & 0 \\ 0 & \frac{\theta_f}{\theta} + \frac{\gamma}{\alpha^2} \end{bmatrix}$ . Notice that  $(A_1, B_1)$  is stabilizable. Therefore,  $\mathcal{F}$  is non-empty. After some elementary calculations we get

$$\mathcal{F} = \left\{ \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix} \mid f_4 \neq \alpha \text{ and } -\delta + \frac{(f_3 - \alpha f_1)(f_4 - \alpha) - (f_4 - \alpha f_2)f_3}{\alpha(\alpha - f_4)} < 0 \right\}.$$

Introducing  $\mu := \frac{1}{\alpha^2 \frac{\theta_f}{\theta} + \gamma}$ ,  $a := \frac{1}{\alpha^2 \theta_l} + \frac{\mu}{\alpha^2}$ ,  $b := -\delta$  and  $c := -\alpha^2$ , (14) becomes  $ak^2 - 2bk - c = 0$ .

Given our parametric assumptions it is easily verified that this equation has a stabilizing solution. Its stabilizing solution is  $k^* := \frac{b + \sqrt{b^2 + ac}}{a}$ . So by Theorem 3.1 all feedback matrices satisfying (15) solve the problem. A simple elaboration of (15) shows that this set of feedback matrices is

$$\mathcal{F}^* := \left\{ \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix} \mid f_1 = -\frac{k^*}{\alpha^2 \theta_l} + \eta f_2, \ f_3 = \frac{k^* \mu}{\alpha} + \eta f_4, \ f_2 \in \mathbb{R}, \ f_4 \in \mathbb{R} \setminus \{\alpha\} \right\}, \quad (27)$$

---

<sup>6</sup>We assume that for a specific application the model parameters are such that this variable will always be nonnegative.

where  $\eta := -\frac{k^*\mu}{\alpha^2}$ . Notice that, by Theorem 3.1, for every  $F \in \mathcal{F}^*$ ,  $A_1 + B_1F(I + X_2B_2F)^{-1}X_1 = A_1 - B_1\bar{R}^{-1}(-B_2^TQ_{12}^T + B_1^TK^*) = b - ak^* < 0$ .

Next we calculate the corresponding optimal "open-loop" control for this problem. That is, if we ignore the potential dynamics of the  $CD$  variable and reduce the optimization problem by substitution of  $CD = \frac{1}{\alpha}u_f$  into the differential equation for  $EW$  and the cost function. This yields that the

problem is equivalent to the minimization of  $J = \int_0^\infty \{[EW \ u_l \ u_f] \begin{bmatrix} -1 & 0 & 0 \\ 0 & \theta_l & 0 \\ 0 & 0 & \frac{\theta_f}{\theta} + \frac{\gamma}{\alpha^2} \end{bmatrix} \begin{bmatrix} EW \\ u_l \\ u_f \end{bmatrix}\} dt$

subject to  $\dot{EW}(t) = -\delta EW(t) + [1 \ \frac{-1}{\alpha}] \begin{bmatrix} u_l(t) \\ u_f(t) \end{bmatrix}$ ,  $EW(0) = x_0$ . The solution to this problem is (see

e.g. [7])  $\begin{bmatrix} u_l \\ u_f \end{bmatrix} = \begin{bmatrix} \frac{-\tilde{k}}{\theta_l} \\ \alpha\tilde{k}\mu \end{bmatrix} EW(t)$ , where  $EW(\cdot)$  solves  $\dot{EW}(t) = (-\delta - \frac{\tilde{k}}{\theta_l} - \mu\tilde{k})EW(t)$  and  $\tilde{k}$  is the

stabilizing solution of the algebraic Riccati equation  $\alpha^2 a\tilde{k}^2 - 2b\tilde{k} - \frac{c}{\alpha^2} = 0$ . It is easily verified from this that  $\tilde{k} = \frac{k^*}{\alpha^2}$ .

In case this "open-loop" control is used to control system (24,25) the next closed-loop dynamics result

$$\begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix} \dot{x}(t) = \begin{bmatrix} -\delta - \frac{\tilde{k}}{\theta_l} & -1 \\ \epsilon\beta - \alpha\tilde{k}\mu & \alpha \end{bmatrix} x(t).$$

Some elementary analysis shows that the eigenvalues of this system will converge to  $-\delta - \frac{\tilde{k}}{\theta_l} - \mu\tilde{k}$  and  $\frac{\alpha}{\epsilon}$  if  $\epsilon$  converges to zero. So, for small  $\epsilon$ , this control will in fact produce a destabilizing control instead of a stabilizing one. Since  $\alpha > 0$  in (25) this point was to be expected from [13].

On the other hand we see that if we use a state feedback control  $[u_l \ u_f]^T = Fx(t)$ , where  $F$  belongs to the set  $\mathcal{F}^*$ , the resulting closed-loop system will be

$$\begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix} \dot{x}(t) = \begin{bmatrix} -\delta + f_1 & -1 + f_2 \\ \epsilon\beta - f_3 & \alpha - f_4 \end{bmatrix} x(t) = \begin{bmatrix} b - ak^* + \eta(f_2 - 1) & f_2 - 1 \\ \epsilon\beta + \eta(\alpha - f_4) & \alpha - f_4 \end{bmatrix} x(t).$$

Introducing  $g := -(b - ak^*)$ ,  $g_2 := f_2 - 1$  and  $g_4 := \alpha - f_4$ , it is easily verified that the eigenvalues of this perturbed system are  $\lambda_{1,2} := \frac{1}{2}\{-g + \eta g_2 + \frac{g_4}{\epsilon} \pm \sqrt{(-g + \eta g_2 + \frac{g_4}{\epsilon})^2 + \frac{4gg_4}{\epsilon} + 4\beta g_2}\}$ . From this it is obvious that this system will be stable for small  $\epsilon$  iff  $g_4 < 0$ , i.e.  $f_4 > \alpha$ . In fact the system will be stable for all  $\epsilon > 0$  if additionally we choose  $f_2$  such that  $\eta(f_2 - 1) < 0$ .

This freedom in choosing  $f_2$  and  $f_4$  can be used to attain some additional goals like, e.g., minimizing the norm of  $F$ . It is easily verified that the norm minimizing  $f_2$  and  $f_4$  are given by  $\frac{k^*}{\alpha^2\theta_l(1+\eta)}$  and  $\frac{-k^*\mu}{\alpha(1+\eta)}$ , respectively. In Figure 1 we plotted two typical responses in case  $\epsilon = 0.1$ ; and the model parameters are  $\delta = 0.1$ ;  $\alpha = 1$ ;  $\beta = 0.05$ ;  $\gamma = 25$ ;  $\theta = 0.005$ ;  $\theta_l = 200$ ;  $\theta_f = 200$  and  $f_4 = 1.1$ . The choice of  $f_4$  is inspired by the fact that  $f_4$  must be larger than 1 and the norm of  $F$  should be small (the norm minimizing value for  $f_4$  is 0.00015), whereas  $f_2 = -0.0293$  minimizes the norm of  $F$ . Figure 1a shows the controls if  $f_2 = -0.0293$  whereas Figure 1b shows these results if  $f_2 = 0.9$ . Figure 1a shows that the norm minimizing control implies a daily inspection that is the first period small, followed by a rapid increase of them and next a steady decline, whereas the long term maintenance cost gradually decline. In particular the daily inspection response differs if a non-norm minimizing feedback is chosen which is illustrated in Figure 1b.  $\square$

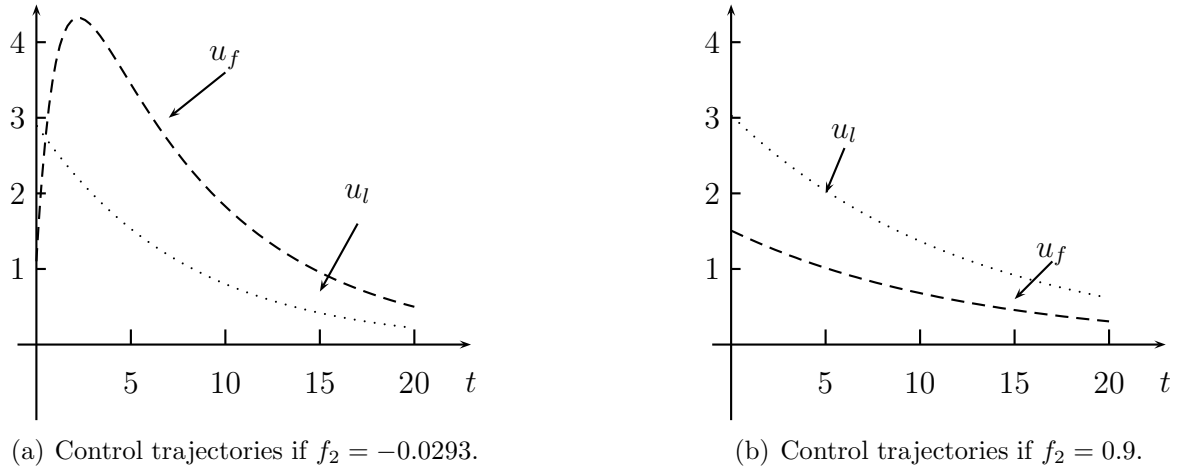


Figure 1: Example 3.4, with  $\epsilon = 0.1$ ,  $f_4 = 1.1$  and  $x(0) = [100 \ 1]^T$ .

**Remark 3.5** 1. Consider  $E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ,  $Q = R = I_2$ . Then straightforward calculations (with  $X = Y = I_2$ ) show that for  $f_{12} \neq 0$ ,  $f_{22} \in \mathbb{R}$ ,  $F = \begin{bmatrix} 0 & f_{12} \\ 1 - \sqrt{2} & f_{22} \end{bmatrix}$  satisfies (14,15). Next consider [19]. It is easily verified that in [19, (10)] matrix  $T_3 = 0$ . Therefore, by [19, (13)]  $L_i = 0$  too. According [19, (7,8)] the set of optimal feedback gain matrices is then parameterized by  $\tilde{F} = \begin{bmatrix} 0 & f_{12} \\ p_0 & 0 \end{bmatrix}$  (with  $p_0 = 1 - \sqrt{2}$ ). Clearly  $F$  and  $\tilde{F}$  only coincide if  $f_{22} = 0$ . This shows that the parametrization given in [19] does not completely specify the set of all optimal feedback gain matrices.

2. In case we choose in Example 3.4 in the model equations  $\delta = 2$ ,  $\alpha = 1$  and  $\epsilon = 0$  (i.e. equation (25) is replaced by  $0 = CD(t) - u_f(t)$ ), and in the objective function  $\gamma = \theta_l = \theta_f = \frac{1}{2}$  and  $\theta = 1$ , the problem has the optimal feedback solution  $F^* = \begin{bmatrix} 1 & 1 \\ -\frac{1}{3} & 0 \end{bmatrix}$  (with  $k^* = -\frac{1}{3}$ ). However, equations (22, 23) do not have this solution  $F^*$ . For, if these equations would have this solution then, since by (23)  $F^* = -R^{-1}B^TK$ , matrix  $K$  should equal  $\frac{-1}{2} \begin{bmatrix} 1 & 1 \\ \frac{1}{3} & 0 \end{bmatrix}$ . But it is easily verified that in that case the equation  $LE = E^TK$  does not have a solution  $L$ . This demonstrates that Proposition 3.3 only provides a sufficient condition for the existence of an optimal solution.  $\square$

**Example 3.6** This model serves to illustrate a case where we do not a priori reduce the number of variables in the model.

Consider a consumer who has to decide about his saving pattern assuming that his labor income will decline over time. His consumption at time  $t$  consists of his labor income, dividends paid by his portfolio minus the cost of investing a part  $u(t)$  in an additional portfolio. The expected long run yield of his portfolio is assumed to be  $r$ . Let  $W(t)$  denote his labor income,  $C(t)$  his consumption,  $S(t)$  the value of his portfolio and  $u(t)$  the change in the portfolio, all at time  $t$ . Assume that the consumer wants to maximize the following utility function, where for simplicity we skipped a

discounting of future utility:

$$J := \int_0^\infty C^2(t) - \bar{\gamma}u^2(t)dt$$

subject to the constraints:

$$\dot{W}(t) = -\alpha W(t), W(0) = W_0; \dot{S}(t) = rS(t) + u(t), x(0) = x_0 \text{ and } C(t) = W(t) + \nu S(t) - u(t).$$

Here  $\nu S$  models the paid long run expected dividends. All parameters are assumed to be positive. The parameter  $\gamma$  can be interpreted as a habit formation parameter, modeling the fact that the consumer is reluctant to change his portfolio. Within our standard framework this means that with

$$x^T(t) := [W(t) S(t) C(t)] \text{ and } E := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ we have to solve the following optimization problem:}$$

$$\min_{u=Fx} \int_0^\infty x^T(t) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} x(t) + \bar{\gamma}u^2(t)dt \text{ subject to } E\dot{x}(t) = \begin{bmatrix} -\alpha & 0 & 0 \\ 0 & r & 0 \\ -1 & -\nu & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(t). \quad (28)$$

$$\text{It is easily verified that with } Y^T := I \text{ and } X := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & \nu & 1 \end{bmatrix}, Y^T E X = E \text{ and } Y^T A X =$$

$$\begin{bmatrix} -\alpha & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\text{Consequently, } B_1 = [0 \ 1]^T, B_2 = 1, A_1 = \begin{bmatrix} -\alpha & 0 \\ 0 & r \end{bmatrix}, X^T Q X = \begin{bmatrix} -1 & -\nu & -1 \\ -\nu & -\nu^2 & -\nu \\ -1 & -\nu & -1 \end{bmatrix} \text{ and } \bar{R} = \bar{\gamma} - 1 =:$$

$\gamma$ .

Furthermore it is easily verified that  $\mathcal{F} = \{[f_1 \ f_2 \ f_3] \mid f_3 \neq -1 \text{ and } x(t) \rightarrow 0 \text{ in } E\dot{x}(t) = (A + BF)x(t)\}$ .

Next consider the algebraic Riccati equation (14). Elementary calculations show that this equation (14) has a stabilizing solution if and only if  $r > \frac{1+\sqrt{\gamma+1}}{\gamma}\nu$ . Assuming that this condition is satisfied its solution is, with  $\beta := \sqrt{(\gamma r - \nu)^2 - (\gamma + 1)\nu^2}$

$$\begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix} := \begin{bmatrix} \frac{-1}{2\alpha\gamma}(\gamma + \frac{\gamma^2(\alpha-r-\nu)^2}{(\gamma\alpha+\beta)^2}) & -\frac{\gamma(r+\nu)+\beta}{\gamma\alpha+\beta} \\ -\frac{\gamma(r+\nu)+\beta}{\gamma\alpha+\beta} & \gamma r - \nu + \beta \end{bmatrix}.$$

Notice that  $k_1 < 0$ ,  $k_2 < 0$  and  $k_3 > 0$ . A simple elaboration of (15) shows that all feedback matrices satisfying

$$\{[f_1 \ f_2 \ f_3] \mid f_1 = -\frac{1+k_2}{\gamma} - \frac{\gamma+1+k_2}{\gamma}f_3, f_2 = -\frac{\nu+k_3}{\gamma} - \frac{(\gamma+1)\nu+k_3}{\gamma}f_3, f_3 \in \mathbb{R} \setminus \{-1\}\}$$

yield a solution for the problem. So in particular  $f_3$  can be chosen such that the control becomes just a function of two out of the three state variables. Another choice for  $f_3$  can be motivated by choosing that value that yields some additional robustness properties for the proposed control. In

this case one might chose  $f_3$  such that the proposed feedback will be close to optimal too in case the expected long run yield of the portfolio is smaller than  $r$ . Assume that  $\bar{r}$  is a conservative estimate for the yield. Then  $f_3$  can be chosen such that  $\|F(r) - F(\bar{r})\|$  is minimal. This problem can simply be rewritten as the minimization of the distance between two lines in  $\mathbb{R}^3$ . Its solution is left as an exercise for the reader.

Finally notice that the utility  $J$  associated with using this feedback control equals  $-[W_0 \ S_0]K \begin{bmatrix} W_0 \\ S_0 \end{bmatrix}$ .

From this we infer that if the initial income of the consumer just consists of his labor income  $W_0$  and he has to decide which amount to invest in a portfolio  $S_0$ , he has to solve the optimization problem

$$\min_{S_0} [W_0 \ S_0]K \begin{bmatrix} W_0 \\ S_0 \end{bmatrix}, \text{ subject to } S_0 + C_0 = W_0, C_0 \geq \bar{C}, \quad (29)$$

where  $\bar{C}$  is some minimum amount of consumption. Some elementary calculations show that the solution for this problem is  $S_0 = \min\{W_0 - \bar{C}, -\frac{k_2}{k_3}W_0\}$ . So, the higher the initial income is the more will be invested in the portfolio. Furthermore, we see that the fraction of initial income that will be invested in the portfolio declines with a factor  $\frac{1}{\gamma}$  if  $\gamma$  grows.  $\square$

## 4 An application to LQ differential games

In this section we use the equivalence result from Theorem 3.1 to characterize feedback Nash equilibria for regular index one systems in infinite-horizon LQ differential games.

The following notation will be used. For an  $N$ -tuple  $\hat{F} = (\hat{F}_1, \dots, \hat{F}_N) \in \Gamma_1 \times \dots \times \Gamma_N$  for given sets  $\Gamma_i$ , we shall write  $\hat{F}_{-i}(\alpha) = (\hat{F}_1, \dots, \hat{F}_{i-1}, \alpha, \hat{F}_{i+1}, \dots, \hat{F}_N)$  with  $\alpha \in \Gamma_i$ .

Consider the cost function of player  $i$  defined by

$$J_i(x_0, F_1, \dots, F_N) = \int_0^\infty \left( x_i^T(t) Q_i x_i(t) + \sum_{j=1}^N u_j^T(t) R_{ij} u_j(t) \right) dt$$

with  $u_j(t) = F_j x_j(t)$ ,  $j = 1, \dots, N$ , and  $x$  satisfying  $E\dot{x}(t) = Ax(t) + \sum_{j=1}^N B_j u_j(t)$ ,  $x(0) = x_0$ . Assume that  $R_{ij} \geq 0$ ,  $i \neq j$ ,  $R_{ii} > 0$  and  $(F_1, \dots, F_N) \in \mathcal{F}_N$ , where

$$\mathcal{F}_N = \left\{ (F_1, \dots, F_N) \mid \begin{array}{l} \text{all finite eigenvalues of } (E, A + \sum_{j=1}^N B_j F_j) \text{ are stable;} \\ (E, A + \sum_{j=1}^N B_j F_j) \\ \text{has index one; and } (E, A + \sum_{j=1, j \neq i}^N B_j F_j) \text{ is regular and has index one } i = 1, \dots, N \end{array} \right\}. \quad (30)$$

This last assumption spoils the rectangular structure of the strategy spaces, i.e. choices of feedback matrices cannot be made independently. However, such a restriction is motivated by the fact that closed-loop stability and avoidance of derivatives in the control actions is usually a common objective. The notion of feedback Nash equilibrium is defined as follows.

**Definition 4.1** An  $N$ -tuple  $F^* = (F_1^*, \dots, F_N^*) \in \mathcal{F}_N$  is called a *feedback Nash equilibrium* if for all  $i = 1, \dots, N$ ,  $J_i(x_0, F^*) \leq J_i(x_0, F_{-i}^*(\alpha))$  for every consistent  $x_0$  and for each matrix  $\alpha$  such that  $F_{-i}^*(\alpha) \in \mathcal{F}_N$ .  $\square$

Consider, for  $F \in \mathcal{F}_N$ , matrices  $Y_i, X_i$  satisfying

$$Y_i^T E X_i = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \text{ and } Y_i^T (A + \sum_{j=1, j \neq i}^N B_j F_j) X_i = \begin{bmatrix} A_i & 0 \\ 0 & I_{n-r} \end{bmatrix}, \quad i = 1, \dots, N. \quad (31)$$

Let  $B_{i1} := [I_r \ 0] Y_i^T B_i$ ,  $B_{i2} := [0 \ I_{n-r}] Y_i^T B_i$ ,  $X_i =: [X_{i1} \ X_{i2}]$ , with  $X_{i1} \in \mathbb{R}^{n \times r}$  and  $X_{i2} \in \mathbb{R}^{n \times (n-r)}$ .

Furthermore, let  $X_i^T (Q_i + \sum_{j=1, j \neq i}^N F_j^T R_{ij} F_j) X_i =: \begin{bmatrix} \tilde{Q}_{i,11} & \tilde{Q}_{i,12} \\ \tilde{Q}_{i,12}^T & \tilde{Q}_{i,22} \end{bmatrix}$  and  $\bar{R}_i := R_{ii} + B_{i2}^T \tilde{Q}_{i,22} B_{i2}$ .

Next, consider the equations:

$$(A_i + B_{i1} \bar{R}_i^{-1} B_{i2}^T \tilde{Q}_{i,12}^T)^T P_i + P_i (A_i + B_{i1} \bar{R}_i^{-1} B_{i2}^T \tilde{Q}_{i,12}^T) - P_i B_{i1} \bar{R}_i^{-1} B_{i1}^T P_i + \tilde{Q}_{i,11} - \tilde{Q}_{i,12} B_{i2} \bar{R}_i^{-1} B_{i2}^T \tilde{Q}_{i,12}^T = 0 \quad (32)$$

$$\sigma(A_i + B_{i1} \bar{R}_i^{-1} B_{i2}^T \tilde{Q}_{i,12}^T - B_{i1} \bar{R}_i^{-1} B_{i1}^T P_i) \subset \mathbb{C}^- \quad (33)$$

$$-\bar{R}_i^{-1} (-B_{i2}^T \tilde{Q}_{i,12}^T + B_{i1}^T P_i) = F_i (I + X_{i2} B_{i2} F_i)^{-1} X_{i1} \quad (34)$$

Theorem 4.2, below, states that all feedback Nash equilibria are determined by the solutions of (31-34).

**Theorem 4.2** Assume  $(E, A)$  is regular and has index one, and  $(E, A, [B_1 \ \dots \ B_N])$  is finite dynamics stabilizable. Let  $(Y_i^*, X_i^*, F_i^*, P_i^*)$ ,  $i = 1, \dots, N$  solve (31-34) and  $\bar{R}_i > 0$ . Then  $(F_1^*, \dots, F_N^*)$  is a feedback Nash equilibrium. Conversely, assume  $Q_i \geq 0^7$ . Then, if  $(F_1^*, \dots, F_N^*)$  is a feedback Nash equilibrium, there exist  $(Y_i, X_i, F_i, P_i)$ ,  $i = 1, \dots, N$ , solving (31-34) with  $F_i = F_i^*$ .

Furthermore, if the game has a feedback Nash equilibrium the with this equilibrium corresponding cost for player  $i$  is

$$J_i(x_0, F^*) = x_{i0}^T P_i x_{i0}, \text{ where } x_{i0} = [I_r \ 0] X_i^{-1} x_0. \quad (35)$$

**Proof:** Let  $(Y_i^*, X_i^*, F_i^*, P_i^*)$ ,  $i = 1, \dots, N$  solve (31-34). Next, consider the minimization by player  $i$  of the cost functional

$$J_i(x_0, F_{-i}^*(F_i)) = \int_0^\infty \left\{ x^T(t) (Q_i + \sum_{j=1, j \neq i}^N F_j^{*T} R_{ij} F_j^* + F_i^T R_{ii} F_i) x(t) \right\} dt, \quad (36)$$

subject to the system

$$E \dot{x}(t) = (A + \sum_{j=1, j \neq i}^N B_j F_j^*) x(t) + B_i F_i x(t), \quad x(0) = x_0. \quad (37)$$

By assumption  $(E, A + \sum_{j=1, j \neq i}^N B_j F_j^*)$  is regular and has index one. Furthermore, by (33),  $(E, A + \sum_{j=1, j \neq i}^N B_j F_j^*, B_i)$  is finite dynamics stabilizable. So, with  $A := A + \sum_{j=1, j \neq i}^N B_j F_j^*$ ;  $B := B_i$ ;  $Q := Q_i + \sum_{j=1, j \neq i}^N F_j^{*T} R_{ij} F_j^*$  and  $R := R_{ii}$  in Theorem 3.1, we conclude that  $F_i^*$  solves the above minimization problem (36,37). The corresponding minimum cost is given by (35). This proves the first part of the claim.

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<sup>7</sup>This assumption is made to ensure that matrix  $\bar{R} > 0$  in Theorem 3.1. Without this assumption one can just conclude that this matrix is semi-positive definite. How to generalize this part of the theorem remains a subject of future research.

Next, assume that  $F^* := (F_1^*, \dots, F_N^*) \in \mathcal{F}_N$  is a feedback Nash equilibrium. Then, by definition,

$$J_i(x_0, F^*) \leq J_i(x_0, F_{-i}^*(F_i))$$

for every consistent  $x_0$  and for each matrix  $F_i$  such that  $F_{-i}^*(F_i) \in \mathcal{F}_N$ . Or, stated differently,  $F_i^*$  solves the minimization problem (36,37). Notice that by definition of  $\mathcal{F}_N$ ,  $(E, A + \sum_{j=1, j \neq i}^N B_j F_j^*)$  is regular and has index one. Furthermore it follows from the fact that  $F^*$  is a Nash equilibrium that  $(E, A + \sum_{j=1, j \neq i}^N B_j F_j^*), B_i)$  is finite dynamics stabilizable (see also footnote 2). Hence, by Theorem 3.1, there exists real symmetric matrices  $(Y_i, X_i, F_i^*, P_i)$ ,  $i = 1, \dots, N$ , solving (31-34).  $\square$

**Example 4.3** To illustrate Theorem 4.2 we reconsider Example 3.4. Assume that an external company is hired who is responsible for the daily operating of the machine. Since the revenues of this company are closely related to those of the owner of the machine, we consider a similar objective function for this company, i.e.

$$J_2 = \int_0^\infty \{\gamma_2 E W^2(t) - \gamma_3 C D^2(t) - r_3 u_l^2(t) - r_4 u_f^2(t)\} dt. \quad (38)$$

Straightforward, though lengthy, calculations show that the set of feedback Nash equilibria for this game are parameterized by  $(s, u)$ , where  $u \neq 0$  and  $s \in \mathbb{R}$  is such that  $w_i(s) \geq 0$ , (see below). They are

$$(F_1^*, F_2^*) = ([a_2 + \delta, 1 - \alpha t], [\frac{s}{u}, \alpha - \frac{1}{u}]). \quad (39)$$

Here, with  $r_2 := \frac{\theta_f}{\theta + (1-\theta)\epsilon}$  and  $w_1 := (\delta + s)^2 + \frac{1}{\theta_l}(-1 + s^2(r_2\alpha^2 + \gamma_1))$ ,  $a_2 = \alpha s t - \sqrt{w_1}$ . Next introduce  $a_1 := -s - \delta$ ,  $e := \alpha^2 r_4 + \gamma_3 + r_3(1 - \alpha t)^2$ ,  $h := a_2 e + \alpha r_3 t(a_2 + \delta)(1 - \alpha t)$  and  $w_2 := h^2 + \alpha^2 t^2((- \gamma_2 + r_3(a_2 + \delta)^2)e - r_3^2(a_2 + \delta)^2(1 - \alpha t)^2)$ . The corresponding matrices  $(Y_i^*, X_i^*, P_i^*)$  solving (31-34) are (with  $a, b, c, d \neq 0$ )

$$Y_1^T = \begin{bmatrix} \frac{1}{a} & \frac{u}{b} \\ 0 & \frac{a}{b} \end{bmatrix}; X_1 = \begin{bmatrix} a & 0 \\ sa & b \end{bmatrix}; Y_2^T = \begin{bmatrix} \frac{1}{c} & \frac{t}{\alpha d} \\ 0 & \frac{1}{\alpha d} \end{bmatrix}; X_2 = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}; P_1 = \theta_l a^2(a_1 + \sqrt{w_1});$$

$$P_2 = \frac{c^2}{\alpha^2 t^2}(h + \sqrt{w_2}); A_1 = a_1; A_2 = a_2 \text{ and } t = \frac{2\theta_l r_3 w_1 - \sqrt{w_1}(r_3(\delta + s) + s(\alpha^2 r_4 + \gamma_3))}{\alpha(s^2(-\theta_l r_4 \alpha^2 - \theta_l \gamma_3 + r_3 \gamma + r_2 r_3 \alpha^2) - r_3 + \theta_l \gamma_2)}.$$

The cost associated with equilibrium (39) are for player one  $\theta_l s_0^2(a_1 + \sqrt{w_1})$  and for player two  $\frac{s_0^2}{\alpha^2 t^2}(h + \sqrt{w_2})$ , where  $s_0 = [1 \ 0]x_0$ .

In Figure 2 we visualized the set of feedback Nash equilibria for some specific choice of parameters. Obviously, players will only participate in the game if their revenues will be positive. Furthermore, solutions that are left from  $(J_1, J_2) \approx (0.5, 0.18)$  are not Pareto efficient. So, this example clearly demonstrates that in general there may exist an infinite number of feasible feedback Nash equilibria. Numerical computations show that the value for which the norm of  $F_1$  is minimal within the feasible region is attained at that point where the revenues of player 2 become zero. Furthermore, elementary calculations show that for a fixed choice of the parameter  $s$ , the norm of  $F_2$  is minimized at  $u = \frac{s^2 + 1}{\alpha}$ . So, assuming additionally that the players both will try to minimize the norm of their feedback gain matrix does not provide a unique equilibrium solution in this case. To arrive at a unique equilibrium solution it seems realistic, given the context of this problem, to assume that players will look for

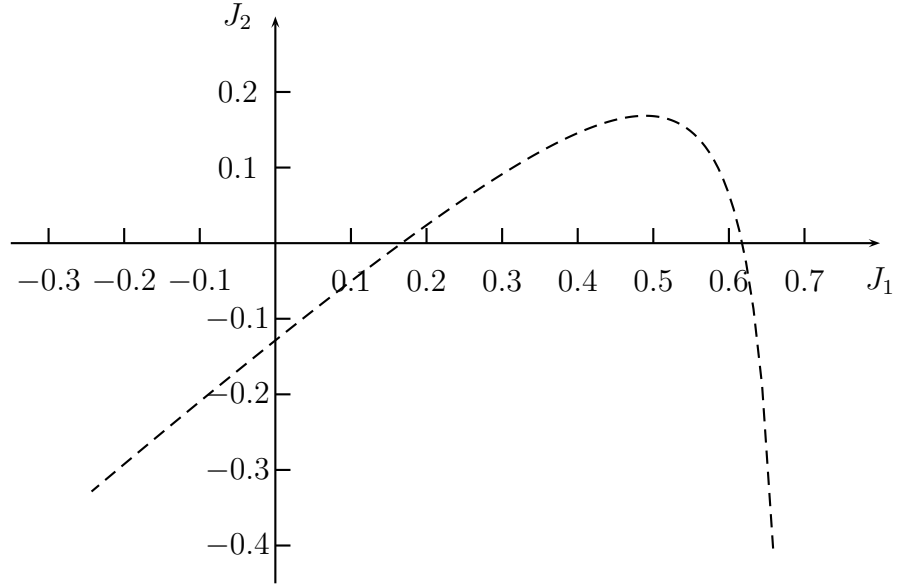


Figure 2: Set of feedback Nash equilibria Example 4.3, with  $\delta = 0.1$ ;  $s_0 = \alpha = 1$ ;  $\gamma = \gamma_3 = r_1 = 2$ ;  $r_2 = r_4 = 10$ ;  $\gamma_2 = 0.9$ ;  $r_3 = 1.5$ .

different criteria involving robustness properties of the strategies w.r.t.  $\epsilon$ . However, this remains a subject for future research.

Finally notice that by choosing different parameters in this model sometimes a unique Pareto efficient equilibrium exists, whereas also parameterizations exist where no solution exists (assuming that both players will only engage if they make profits).  $\square$

## 5 Concluding Remarks

In this note we considered the problem to find a static stabilizing state feedback controller for an index one descriptor system. We derived both necessary and sufficient conditions for the existence of such a controller. In general this controller is not uniquely determined. We illustrated this in an example. The solvability conditions are formulated in terms of a transformed system. A sufficient condition in terms of the original model parameters was provided too.

The equivalence result has been used to formulate both necessary and sufficient conditions for the existence of a linear state feedback Nash equilibrium in a differential game setting. This under the assumption that the players have perfect state feedback information.

Since the optimal solution is usually not uniquely determined the question arises how this freedom can be exploited to select amongst these optimal solutions one that has some additional nice properties (like e.g. robustness or minimum norm as we did in our example (see e.g. [1])). This characterization could then be exploited too to derive a numerical efficient procedure for calculating such a solution. Moreover this property could be useful to formulate a refinement of the Nash equilibrium concept. This remains a topic for future research. Another open problem is how to generalize these results for descriptor systems that have a higher order index.



## 6 Appendix: Proof of Proposition 3.3

With  $\tilde{K} := Y^{-1}KX$ ,  $\tilde{L} := X^TLY^{-T}$  and  $\tilde{x}(t) := X^{-1}x(t)$  (22,23) can be rewritten as

$$\tilde{L} \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} A_1^T & 0 \\ 0 & I \end{bmatrix} \tilde{K} + X^T Q X - \tilde{L} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} R^{-1} [B_1^T \ B_2^T] \tilde{K} = 0; \quad (40)$$

$$\tilde{L} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \tilde{K} \quad (41)$$

and

$$u(t) = -R^{-1} [B_1^T \ B_2^T] \tilde{K} \tilde{x}(t), \text{ where } \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \dot{\tilde{x}}(t) = \left( \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} R^{-1} [B_1^T \ B_2^T] \tilde{K} \right) \tilde{x}(t).$$

Now, assume there exist matrices  $\tilde{L}$  and  $\tilde{K}$  satisfying the above relationships. Let  $\tilde{L} =: \begin{bmatrix} L_1 & L_2 \\ L_4 & L_3 \end{bmatrix}$

and  $\tilde{K} =: \begin{bmatrix} K_1 & K_4 \\ K_2 & K_3 \end{bmatrix}$ . Then it follows directly from (41) that  $K_1 = L_1$  and  $L_4 = K_4 = 0$ .

Next notice that  $F$  in (23) satisfies  $F = -R^{-1} [B_1^T \ B_2^T] \tilde{K} X^{-1}$ . From this it follows immediately that

$$RF X_1 = -(B_1^T L_1 + B_2^T K_2) \text{ and } RF X_2 = -B_2^T K_3. \quad (42)$$

Using (42), a simple expansion of (40) shows that the matrices  $L_i$  and  $K_i$  satisfy

$$0 = L_1 A_1 + A_1^T L_1 + X_1^T Q X_1 + (L_1 B_1 + L_2 B_2) F X_1 \quad (43)$$

$$0 = L_2 + X_1^T Q X_2 + (L_1 B_1 + L_2 B_2) F X_2 \quad (44)$$

$$0 = K_2 + X_2^T Q X_1 + L_3 B_2 F X_1 \quad (45)$$

$$0 = L_3 + K_3 + X_2^T Q X_2 + L_3 B_2 F X_2. \quad (46)$$

Next introduce, consistent with the previous notation,  $G := I + B_2 F X_2$ ;  $\tilde{G} := I + X_2 B_2 F$ ; and  $\hat{G} := I + F X_2 B_2$ . Then, by Lemma 2.3,

$$X_2 G^{-1} = \tilde{G}^{-1} X_2 \text{ and } X_2 B_2 \hat{G}^{-1} = \tilde{G}^{-1} X_2 B_2. \quad (47)$$

Then post-multiplying (44) by  $B_2$  we obtain  $L_2 B_2 \hat{G} = -(X_1^T Q + L_1 B_1 F) X_2 B_2$ . Or, using (47),

$$L_2 B_2 = -(X_1^T Q + L_1 B_1 F) X_2 B_2 \hat{G}^{-1} = -(X_1^T Q + L_1 B_1 F) \tilde{G}^{-1} X_2 B_2. \quad (48)$$

Using (42), pre-multiplication of (46) by  $B_2^T$  gives  $B_2^T L_3 G = -B_2^T K_3 - B_2^T X_2^T Q X_2 = (RF - B_2^T X_2^T Q) X_2$ . Therefore, using (47), we get

$$B_2^T L_3 = (RF - B_2^T X_2^T Q) X_2 G^{-1} = (RF - B_2^T X_2^T Q) \tilde{G}^{-1} X_2. \quad (49)$$

Pre-multiplication of (45) by  $B_2^T$  yields  $B_2^T K_2 = -B_2^T X_2^T Q X_1 - B_2^T L_3 B_2 F X_1$ . So, using (49),

$$\begin{aligned} B_2^T K_2 &= -B_2^T X_2^T Q X_1 - (RF - B_2^T X_2^T Q) \tilde{G}^{-1} X_2 B_2 F X_1 \\ &= -B_2^T X_2^T Q X_1 - (RF - B_2^T X_2^T Q) \tilde{G}^{-1} (-I + I + X_2 B_2 F) X_1 \\ &= (RF - B_2^T X_2^T Q) \tilde{G}^{-1} X_1 - RF X_1. \end{aligned} \quad (50)$$

Using (50) we conclude next from (42) that

$$\begin{aligned} -B_2^T Q_{12}^T + B_1^T L_1 &= -B_2^T Q_{12}^T - (RF X_1 + B_2^T K_2) \\ &= -B_2^T X_2^T Q \tilde{G} \tilde{G}^{-1} X_1 - (RF - B_2^T X_2^T Q) \tilde{G}^{-1} X_1 = -\bar{R} F \tilde{G}^{-1} X_1. \end{aligned} \quad (51)$$

Introducing  $P := (A_1 + B_1 \bar{R}^{-1} B_2^T Q_{12}^T)^T L_1 + L_1 (A_1 + B_1 \bar{R}^{-1} B_2^T Q_{12}^T) - L_1 B_1 \bar{R}^{-1} B_1^T L_1 + Q_{11}$ , we obtain then from (43), using (48) and (51) respectively, that  $L_1$  satisfies

$$\begin{aligned} 0 &= A_1^T L_1 + L_1 A + X_1^T Q X_1 + (L_1 B_1 + L_2 B_2) F X_1 \\ &= P - Q_{12} B_2 \bar{R}^{-1} B_1^T L_1 - L_1 B_1 \bar{R}^{-1} B_2^T Q_{12}^T + L_1 B_1 \bar{R}^{-1} B_1^T L_1 + L_1 B_1 F X_1 \\ &\quad - (X_1^T Q + L_1 B_1 F) \tilde{G}^{-1} X_2 B_2 F X_1 \\ &= P - Q_{12} B_2 \bar{R}^{-1} B_1^T L_1 - L_1 B_1 \bar{R}^{-1} B_2^T Q_{12}^T + L_1 B_1 \bar{R}^{-1} B_1^T L_1 \\ &\quad + L_1 B_1 F (I + X_2 B_2 F) \tilde{G}^{-1} X_1 - (X_1^T Q + L_1 B_1 F) X_2 B_2 F \tilde{G}^{-1} X_1 \\ &= P + (Q_{12} B_2 - L_1 B_1) \bar{R}^{-1} (B_2^T Q_{12}^T - \bar{R} F \tilde{G}^{-1} X_1 - B_1^T L) - Q_{12} B_2 \bar{R}^{-1} B_2^T Q_{12}^T \\ &= P - Q_{12} B_2 \bar{R}^{-1} B_2^T Q_{12}^T. \end{aligned}$$

Since by assumption  $L_1$  is symmetric, clearly  $L_1$  and  $F$  satisfy (14,15).

So by Theorem 3.1 the cost are  $x_1^T L_1 x_1$ . To see that these correspond with  $x_0^T L E x_0$  we first rewrite the cost as follows:

$$\begin{aligned} x_1^T L_1 x_1 &= x_0^T X^{-T} \begin{bmatrix} I \\ 0 \end{bmatrix} [I \ 0] X^T L Y^{-T} \begin{bmatrix} I \\ 0 \end{bmatrix} [I \ 0] X^{-1} x_0 \\ &= x_0^T X^{-T} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} X^T L Y^{-T} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} X^{-1} x_0 = x_0^T X^{-T} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} X^T L E x_0. \end{aligned}$$

Next notice that since  $Y^T E X = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ ,  $E X_2 = 0$ . Since, moreover,  $L_4 = 0$  it follows that

$$0 = [0 \ I] X^T L Y^{-T} \begin{bmatrix} I \\ 0 \end{bmatrix} = X_2^T L Y^{-T} Y^T E X \begin{bmatrix} I \\ 0 \end{bmatrix} = X_2^T L E X_1.$$

So, combining both results we have that  $X_2^T L E X = 0$  or, equivalently,  $X_2^T L E = 0$ .

Now,  $X^{-T} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} X^T L E = L E$  if and only if  $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} X^T L E = X^T L E$ . Clearly the last equality holds if and only if  $X_2^T L E = 0$ , which was shown above to be the case.  $\square$

## References

- [1] Bender, D.J., & Laub, A.J., (1987). The linear-quadratic optimal regulator for descriptor systems. *I.E.E.E. Transactions on Automatic Control*, 32, 672-688.
- [2] Benner, P., Byers, R., Mehrmann, V., & Xu, H., (1999). Numerical solution of linear-quadratic control problems for descriptor systems. *Proc. IEEE conf. Computer Aided Control Systems Design, Hawaii*, CDROM.

- [3] Broek, W.A. van den, Engwerda, J.C., & Schumacher, J.M., (2003). An equivalence result in linear quadratic control theory. *Automatica*, 39, 355-359.
- [4] De Carlo, R.A., & Saeks, R., (1981). *Interconnected Dynamical Systems*, Marcel Dekker, New York.
- [5] Cheng, Z., Hong, H., & Zhang, J., (1988). The optimal regulation of generalized spate-space systems with quadratic cost. *Automatica*, 24, 679-710.
- [6] Cobb D., (1983). Descriptor variable systems and optimal state regulation. *I.E.E.E. Transactions on Automatic Control*, 28, 601-611.
- [7] Engwerda J.C., (2005). *Linear Quadratic Dynamic Optimization and Differential Game Theory*, John Wiley & Sons, Chichester.
- [8] Engwerda, J.C., & Salmah, (2008). The open-loop linear quadratic differential game for index one descriptor systems. To appear in *Automatica*.
- [9] Gantmacher, F., (1959). *Theory of Matrices*, vol. I,II. Chelsea: New York.
- [10] Geerts, T., (1994). Regularity and singularity in linear-quadratic control subject to implicit continuous-time systems. *Circuits, System Signal Process.*, 13, 19-30.
- [11] Kalogeropoulos, G., & Arvanitis, K.G., (1998). A matrix-pencil-based interpretation of inconsistent initial conditions and system properties of generalized state-space. *IMA Journal of Mathematical Control & Information*, 15, 73-91.
- [12] Katayama, T., & Minamino, K., (1992). Linear quadratic regulator and spectral factorization for continuous-time descriptor systems. *Proc. 31<sup>th</sup> IEEE Conf. Decision Control, Tucson, Arizona*, 967-972.
- [13] Kokotovic, P.V., O'Malley Jr., R.E., & Sannuti, P., (1976). Singular perturbations and order reduction in control theory - an overview. *Automatica*, 12, 123-132.
- [14] Kunkel, P., & Mehrmann, V., (1997). The linear quadratic control problem for linear descriptor systems with variable coefficients. *Math. Control, Signals, Systems*, 10, 247-264.
- [15] Mehrmann V.L., (1991). *The Autonomous Linear Quadratic Control Problem*. In: *Lecture Notes in Control and Information Sciences* (Eds. M. Thoma and A. Wyner), Vol.163. Springer-Verlag: Berlin.
- [16] Pandolfi, L., (1981). On the regulator problem for linear degenerate control systems. *Journal of Optimization Theory and Applications*, 33, 241-254.
- [17] Salmah, (2006). Kontrol Optimal Sistem Regulator Deskriptor Untuk Permainan Dinamis (Optimal Control of Regulator Descriptor Systems for Dynamic Games). Dissertation, Universitas Gadjah Mada, Yogyakarta, Indonesia.
- [18] Takaba, K., & Katayama, T. (1998).  $H^2$  Output feedback control for descriptor systems. *Automatica*, 34, 841-850.

- [19] Wang, Y-Y, Frank, P.M., & Clements D.J., (1993). The robustness properties of the linear quadratic regulators for singular systems. *I.E.E.E. Transactions on Automatic Control*, 38, 96-100.
- [20] Xu, H., & Mizukami, K., (1993). Two-person two-criteria decision making problems for descriptor sytems. *JOTA*, 78, 163-173.
- [21] Xu, H., & Mizukami, K., (1994). On the Isaacs equation of differential games for descriptor systems. *JOTA*, 83, 405-419.
- [22] Zhu, J., Ma, S.,& Cheng Z., (2002). Singular LQ problem for nonregular descriptor systems. *I.E.E.E. Transactions on Automatic Control*, 47, 1128-1133.